

A Certain Family of Continued Fractions I

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Abstract

We discuss a certain family of continued fractions which contains the usual continued fractions, the negative continued fractions and the shortest continued fractions. This family is defined with one parameter. We give the reduced sets for quadratic irrational numbers.

1. Introduction. To give a solution for Eisenstein's problem, the author used three kinds of continued fraction expansions [1,2,3]. First of all we will illustrate them. Let ω be a real irrational number. The first is the usual (positive) continued fraction expansion (CFE) of ω , that is a sequence of real irrational numbers $\omega_0, \omega_1, \omega_2, \dots$ with

$$\omega_0 = \omega, a_k = [\omega_k], \omega_{k+1} = 1/(\omega_k - a_k) \quad (k = 0, 1, 2, \dots),$$

the second is the negative CFE of $\omega : \omega_0, \omega_1, \omega_2, \dots$ with

$$\omega_0 = \omega, b_k = [\omega_k], \omega_{k+1} = 1/(b_k + 1 - \omega_k) \quad (k = 0, 1, 2, \dots),$$

and the third is the shortest CFE of $\omega : \omega_0, \omega_1, \omega_2, \dots$ with

$$\omega_0 = \omega, c_k = [\omega_k],$$

$$\omega_{k+1} = \begin{cases} 1/(\omega_k - c_k) & (\omega_k - c_k < \frac{1}{2}) \\ 1/(c_k + 1 - \omega_k) & (\omega_k - c_k > \frac{1}{2}) \end{cases} \quad (k = 0, 1, 2, \dots),$$

Received September 28, 2000. Accepted December 7, 2000.

where $[x]$ means the largest integer not greater than x .

We see that the above expansions are all periodic after a certain stage when ω is quadratic. Moreover, for each non-square positive integer D , there is a finite set R_D (called a reduced set for D) such that $\omega_n \in R_D$ for some n whenever ω belongs to D .

In this article we define a family of continued fractions which contains the above fractions and determine the reduced sets for a part of the family. This family is defined with one parameter.

2. Definition. Let S be the set of all irrational numbers and let λ be a real number so that $0 \leq \lambda \leq 1$. We define a mapping $\rho = \rho_\lambda$ of S into itself:

$$\rho(\omega) = \frac{1}{|\omega - c|} \text{ with } c = [\omega + \lambda].$$

Then we have a sequence $\omega, \rho(\omega), \rho^2(\omega), \rho^3(\omega), \dots$. Such continued fractions are called λ -continued fractions. If we take $\lambda = 0$, then we have the usual case. So the usual continued fractions are 0-continued fractions. Similarly, we have the negative case and the shortest case if we take $\lambda = 1$ and $\lambda = 1/2$ respectively.

The mapping ρ_λ can be extended to a mapping of $S \times S$ into itself as follows: for $(\alpha, \beta) \in S \times S$ we define

$$\rho_\lambda((\alpha, \beta)) = \begin{cases} (1/(\alpha - c), 1/(\beta - c)) & \text{if } \alpha > c \\ (1/(c - \alpha), 1/(c - \beta)) & \text{if } \alpha < c \end{cases}$$

where $c = [\alpha + \lambda]$. We have

Proposition 1. $\rho_\lambda(F) \subset F$, where $F = F^+ \cup F^-$, and

$$F^+ = \{(\alpha, \beta) \in S \times S \mid \alpha \geq 1/\lambda, 0 < \beta < 1\},$$

$$F^- = \{(\alpha, \beta) \in S \times S \mid \alpha > 1/(1 - \lambda), -1 < \beta < 0\}.$$

Proof. Put $(\alpha', \beta') = \rho((\alpha, \beta))$ and $c = [\alpha + \lambda]$. Then we have

$$(1) \alpha' = 1/(c - \alpha), \beta' = 1/(c - \beta) \text{ if } c - \lambda \leq \alpha < c,$$

$$(2) \alpha' = 1/(\alpha - c), \beta' = 1/(\beta - c) \text{ if } c < \alpha < c + 1 - \lambda.$$

First we assume that $(\alpha, \beta) \in F^+$. Then we see $\alpha + \lambda \geq 1/\lambda + \lambda \geq 2$, so $c \geq 2$. In Case (1) we have $\alpha' \geq 1/\lambda$ and $c - \beta \geq 2 - \beta > 1$. Hence $0 < \beta' < 1$. This implies that $(\alpha', \beta') \in F^+$. In Case (2) we have $\alpha' > 1/(1-\lambda)$ and $\beta - c \leq \beta - 2 < -1$. Thus $-1 < \beta' < 0$ and $(\alpha', \beta') \in F^-$. Secondly we assume that $(\alpha, \beta) \in F^-$. In Case (1) we have $\alpha' \geq 1/\lambda$ and $c - \beta \geq 1 - \beta > 1$. Hence $0 < \beta' < 1$. This implies that $(\alpha', \beta') \in F^+$. In Case (2) we have $\alpha' > 1/(1-\lambda)$ and $\beta - c \leq \beta - 1 < -1$. Hence $-1 < \beta' < 0$. \square

Remark 1. We note that F^+ or F^- is empty when $\lambda = 0$ or $\lambda = 1$ respectively.

3. Quadratic numbers. By Q we denote the set of all quadratic irrational numbers. Each quadratic irrational number ω can be uniquely expressed as

$$\omega = \frac{b + \sqrt{D}}{a}$$

with integers a, b and a positive non-square integer D such that

$$b^2 \equiv D \pmod{a} \text{ and } \gcd(a, b, (b^2 - D)/a) = 1,$$

and then we say that ω belongs to D . By $Q(D)$ we denote the set of such numbers.

Lemma. If ω is quadratic irrational, then ω and $\rho_\lambda(\omega)$ belong to the same D .

Proof. Assume that ω belongs to D and write $\omega = \frac{b + \sqrt{D}}{a}$ as in the expression we described above. Put $c = [\omega + \lambda]$. Then we have $\rho_\lambda(\omega) = \frac{b' + \sqrt{D}}{a'}$ with $b' = ac - b$ and $a' = \pm(b^2 - D)/a$. If a prime p divides $a', b', (b'^2 - D)/a'$, then p divides a , and hence b . We see that $(b^2 - D)/a = ac^2 - 2cb' \pm a'$ is divided by p , which is a contradiction. \square

Proposition 2. Let D be a positive non-square integer and put

$$F(D) = \{\omega \in Q(D) \mid (\omega, \bar{\omega}) \in F\},$$

where $\bar{\omega}$ denotes the conjugate of ω and F is in Proposition 1. Then $F(D)$ is finite.

Proof. Take $\omega = \frac{b + \sqrt{D}}{a} \in F(D)$. It can be seen that $a, b > 0$ because $-1 < \bar{\omega} < 1 < \omega$. We have $a - \sqrt{D} < b < a + \sqrt{D}$ since $\bar{\omega} < 1 < \omega$. Putting $b = a + r$ with $|r| < \sqrt{D}$, we see that $r^2 \equiv D \pmod{a}$ since $b^2 \equiv D \pmod{a}$. This implies that $a \leq D - r^2 \leq D$. Thus the number of a 's and hence the number of b 's are finite. \square

Remark. We note that for every quadratic irrational number ω belonging to D there is a positive integer n such that $\rho_\lambda^n(\omega) \in F(D)$.

4. Reduced sets. It is well known that the usual continued fraction expansions of quadratic irrational numbers have the reduced set R_0 , that is, the set of purely periodic quadratic irrational numbers. In fact we know (see Appendix) that

$$R_0 = \{\omega \in Q \mid -1 < \bar{\omega} < 0, 1 < \omega\}.$$

Also we know the reduced sets for $\lambda = 1, 1/2[1]$ (see Appendix):

$$R_1 = \{\omega \in Q \mid 0 < \bar{\omega} < 1, 1 < \omega\}.$$

$$R_{1/2} = \{\omega \in Q \mid \frac{1 - \sqrt{5}}{2} < \bar{\omega} < \frac{3 - \sqrt{5}}{2}, 2 < \omega\}.$$

It is known that the set $R_\lambda \cap Q(D)$ is finite for $\lambda = 0, 1, 0.5$. By Proposition 2 we see that the set $R_\lambda \cap Q(D)$ is finite for any λ if R_λ exists and is contained in $F(D)$. From now on we will neglect all elements in $Q \cap Q(\lambda)$, where Q denotes the rational number field.

Theorem 1. Let $0 \leq \lambda < \zeta = \frac{3 - \sqrt{5}}{2} = 0.381966\cdots$, Then ρ_λ is a transformation on the set $R_\lambda = A \cup B \cup C$, where

$$A = \{\omega \in Q \mid 0 < \bar{\omega} < \frac{1}{2}, \frac{1}{\lambda} < \omega\},$$

$$B = \{\omega \in Q \mid -\frac{1}{2} < \bar{\omega} < 0, \frac{1}{1-\lambda} < \omega\},$$

$$C = \{\omega \in Q \mid -1 < \bar{\omega} < -\frac{1}{2}, \frac{1}{1-\lambda} < \omega < \frac{1-\lambda}{\lambda}\}.$$

Proof. Let us prove that $\rho_\lambda(R_\lambda) \subset R_\lambda$. Take $\omega \in R_\lambda$ and put $\alpha = \rho_\lambda(\omega)$ and $c = [\omega + \lambda]$. First assume that $c - \lambda < \omega < c$. Hence $c \geq 2$. We have $\alpha > 1/\lambda$ and $\bar{\alpha} = 1/(c - \bar{\omega})$. If $c \geq 3$, then we see $c - \bar{\omega} \geq 3 - 1/2$, so $0 < \bar{\alpha} < 2/5 < 1/2$. Thus $\alpha \in A$. If $c = 2$, then we have $\omega < 2 < 1/\lambda$, which implies that $\omega \notin A$. Thus $\bar{\omega} < 0$, and $c - \bar{\omega} > 2$. Hence $0 < \bar{\alpha} < 1/2$, so $\alpha \in A$. Secondly we assume that $c < \omega < c + 1 - \lambda$. Then we have $\alpha > 1/(1 - \lambda)$ and $\bar{\alpha} = 1/(\bar{\omega} - c)$. If $c \geq 3$, then $\bar{\omega} - c \leq -5/2$, so $0 > \bar{\alpha} > -2/5 > -1/2$. Thus $\alpha \in B$. If $c = 2$, then we have $2 < \omega < 3 - \lambda < 1/\lambda$ because $0 \leq \lambda < \zeta$. Hence $\omega \notin A$. Then we see that $\bar{\omega} < 0$ and $-1/2 < \bar{\alpha} < 0$. This implies $\alpha \in B$. If $c = 1$, then we see that $\omega < 2 - \lambda < 1/\lambda$, so $\omega \notin A$. Then we have $\omega > 1/(1 - \lambda)$ and $-1 < \bar{\omega} < 0$. Hence we see that $\alpha < (1 - \lambda)/\lambda$ and $-1 < \bar{\alpha} < -1/2$, which implies that $\alpha \in C$.

We prove that ρ_λ is an injection, which implies that ρ_λ is bijective since $R_\lambda \cap Q(D)$ is finite for each D . Suppose that $\rho_\lambda(\omega) = \rho_\lambda(\omega_1)$ for $\omega, \omega_1 \in R_\lambda$. Hence $\omega, \omega_1, \rho_\lambda(\omega)$ are in the same $Q(D)$. Put $c = [\omega + \lambda]$, $c_1 = [\omega_1 + \lambda]$. We have four cases:

- (1) $-\lambda < \omega - c < 0, -\lambda < \omega_1 - c_1 < 0$
- (2) $-\lambda < \omega - c < 0, 0 < \omega_1 - c_1 < 1 - \lambda$
- (3) $0 < \omega - c < 1 - \lambda, -\lambda < \omega_1 - c_1 < 0$
- (4) $0 < \omega - c < 1 - \lambda, 0 < \omega_1 - c_1 < 1 - \lambda$

In Cases (1) and (4), we have $c - c_1 = \omega - \omega_1$, so $c - c_1 = \bar{\omega} - \bar{\omega}_1$. Thus $|c - c_1| = 0, 1$. If $c = c_1$, then we have $\omega = \omega_1$. If $c - c_1 = 1$, then $\bar{\omega} - \bar{\omega}_1 = 1$, so $\bar{\omega} > 0$ and $\bar{\omega}_1 < -1/2$. Hence $\omega > 1/\lambda$ and $\omega_1 < (1 - \lambda)/\lambda$, which implies that $\omega - \omega_1 > 1 = c - c_1$, a contradiction. The same argument works for $c - c_1 = -1$. In Cases (2) and (3), we have $c - \omega = \omega_1 - c_1$ and $2 \leq c + c_1 = \bar{\omega} + \bar{\omega}_1 < 1$, a contradiction. \square

Theorem 2. Let $\zeta \leq \lambda \leq \frac{1}{2}$. Then ρ_λ is a transformation on the set $R_\lambda = A \cup B \cup C \cup D$, where

$$\begin{aligned} A &= \{\omega \in Q \mid \zeta < \bar{\omega} < \frac{1}{2}, \frac{1-\lambda}{1-2\lambda} < \omega\}, \\ B &= \{\omega \in Q \mid 0 < \bar{\omega} < \zeta, \frac{1}{\lambda} < \omega\}, \\ C &= \{\omega \in Q \mid -\frac{1}{2} < \bar{\omega} < 0, \frac{1}{1-\lambda} < \omega\}, \\ D &= \{\omega \in Q \mid \zeta - 1 < \bar{\omega} < -\frac{1}{2}, \frac{1}{1-\lambda} < \omega < \frac{\lambda}{1-2\lambda}\}. \end{aligned}$$

Proof. We prove that $\rho_\lambda(R_\lambda) \subset R_\lambda$. Take $\omega \in R_\lambda$ and put $\alpha = \rho_\lambda(\omega)$ and $c = [\omega + \lambda]$. Clearly we have $\omega + \lambda > 1/(1-\lambda) + \lambda \geq 2$ since $\zeta \leq \lambda$. Thus $c \geq 2$. Note that $2 \leq 1/\lambda \leq 3 - \lambda \leq (1-\lambda)/(1-2\lambda)$.

(1) First suppose $c = 2$. (i) If $\omega < 2$, then we have $1/(1-\lambda) < \omega < 2$ and so $\zeta - 1 < \bar{\omega} < 0$. Hence we see that $\alpha = 1/(2-\omega) > (1-\lambda)/(1-2\lambda)$ and $1/2 > \bar{\alpha} > \zeta$, so $\alpha \in A$. (ii) If $2 < \omega < 1/\lambda$, then we have $\bar{\omega} < 0$, so $\alpha = 1/(\omega-2) > \lambda/(1-2\lambda)$ and $-1/2 < \bar{\alpha} < 0$. Thus $\alpha \in C$. (iii) If $1/\lambda < \omega < 3-\lambda$, then we have $\bar{\omega} < \zeta$. Hence $1/(1-\lambda) < \alpha < \lambda/(1-2\lambda)$ and $\zeta - 1 < \bar{\alpha} < 0$. Thus $\alpha \in C \cup D$.

(2) Secondly suppose $c = 3$. (i) If $3-\lambda < \omega < 3$ and $\omega < (1-\lambda)/(1-2\lambda)$, then $\bar{\omega} < \zeta$. Hence $\alpha = 1/(3-\omega) > 1/\lambda$ and $0 < \bar{\alpha} < \zeta$, which implies $\alpha \in B$. (ii) If $(1-\lambda)/(1-2\lambda) < \omega < 3$, then we have $-1/2 < \bar{\omega} < 1/2$. Hence we see that $\alpha = 1/(3-\omega) > (1-2\lambda)/(2-5\lambda) \geq (1-\lambda)/(1-2\lambda)$ and $0 < \bar{\alpha} < 1/2$, which implies $\alpha \in A \cup B$. (iii) If $3 < \omega < 4-\lambda$, then we can easily see that $\alpha = 1/(\omega-3) > 1/(1-\lambda)$ and $-1/2 < \bar{\alpha} < 0$, which implies $\alpha \in C$.

(3) Finally suppose $c \geq 4$. (i) If $c-\lambda < \omega < c$, then we have $\alpha = 1/(c-\omega) > 1/\lambda$ and $c-\bar{\omega} \geq 4-1/2 = 7/2$, so $0 < \bar{\alpha} < 2/7 < \zeta$, thus $\alpha \in B$. (ii) If $c < \omega < c+1-\lambda$, then we have $\alpha = 1/(\omega-c) > 1/(1-\lambda)$ and $\bar{\omega}-c \leq -7/2$, so $-1/2 < \bar{\alpha} < 0$. Thus $\alpha \in C$.

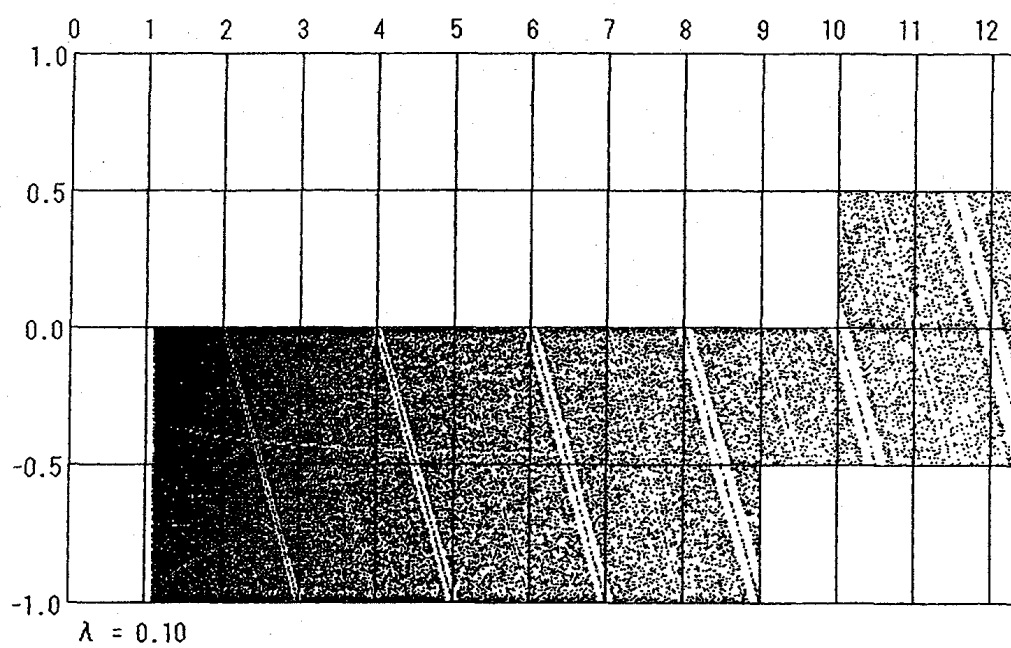
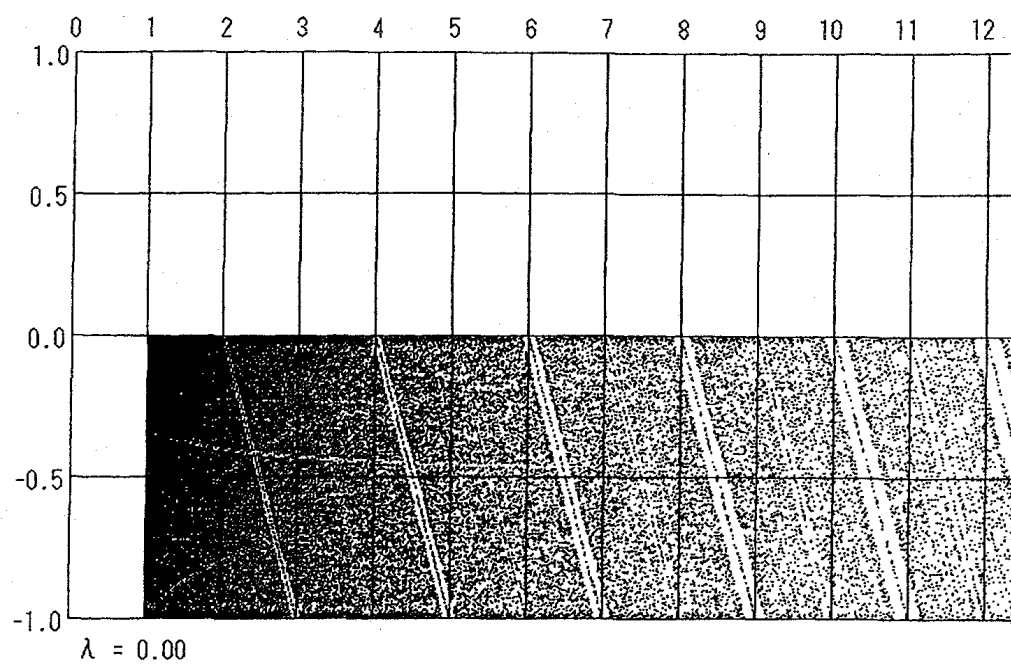
The injectivity of ρ_λ can be shown in a similar way as in our proof of Theorem

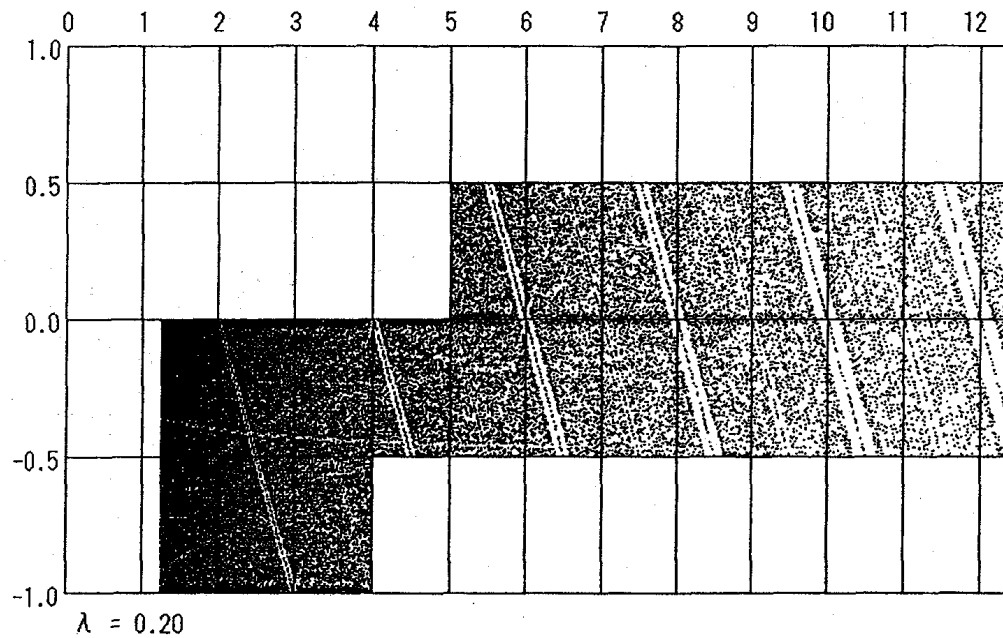
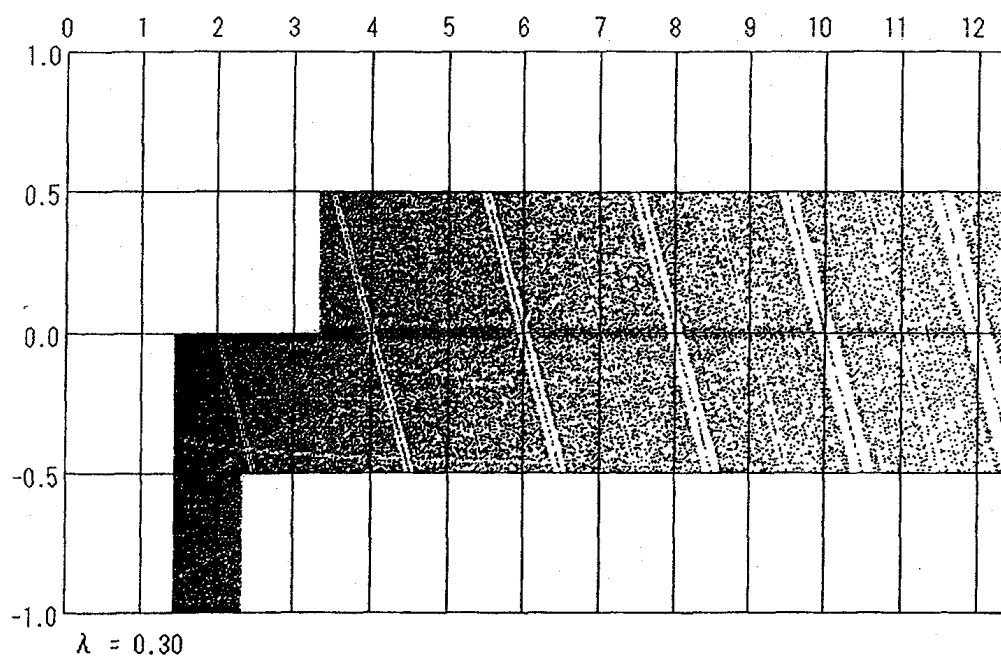
1. \square

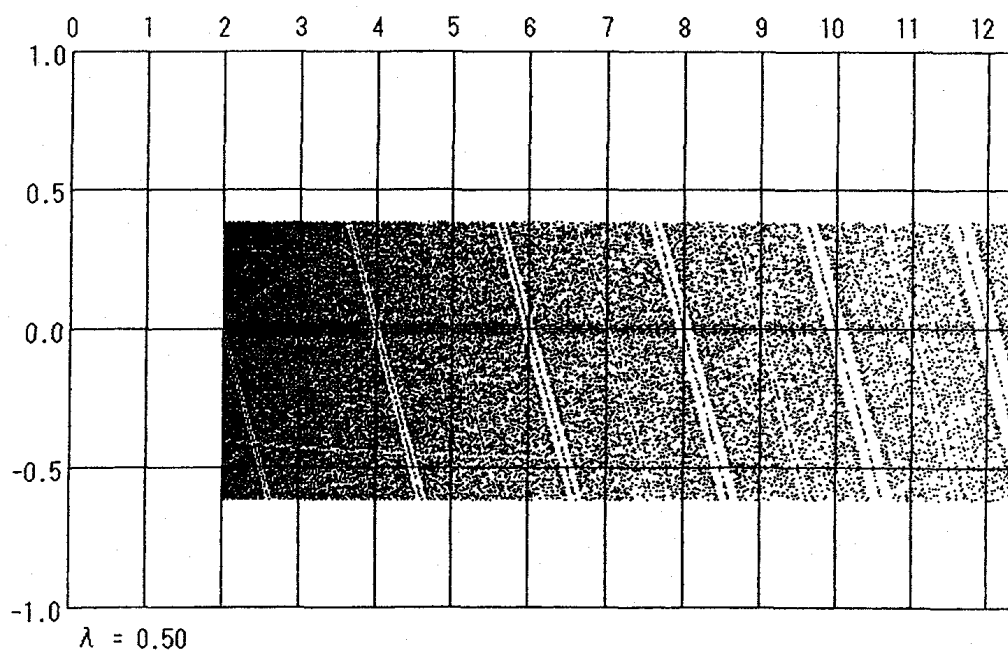
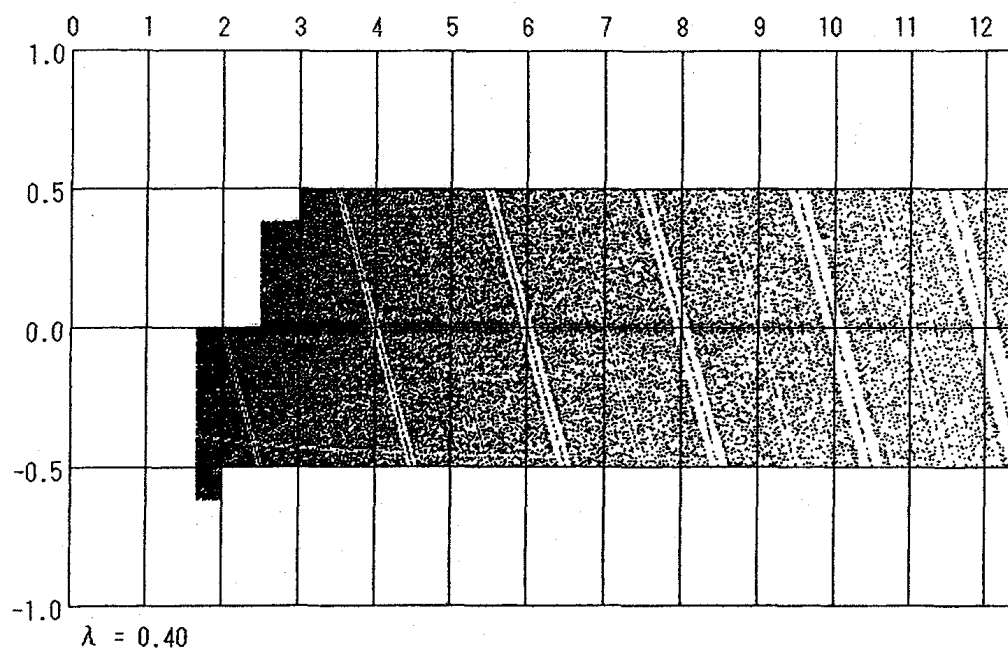
We can see that the finite set $F(D) \cap R_\lambda$ is partitioned into a number of cycles when $0 \leq \lambda \leq 1/2$ from Theorems 1 and 2. In other words, each irrational quadratic number belonging to D in R_λ is purely periodic.

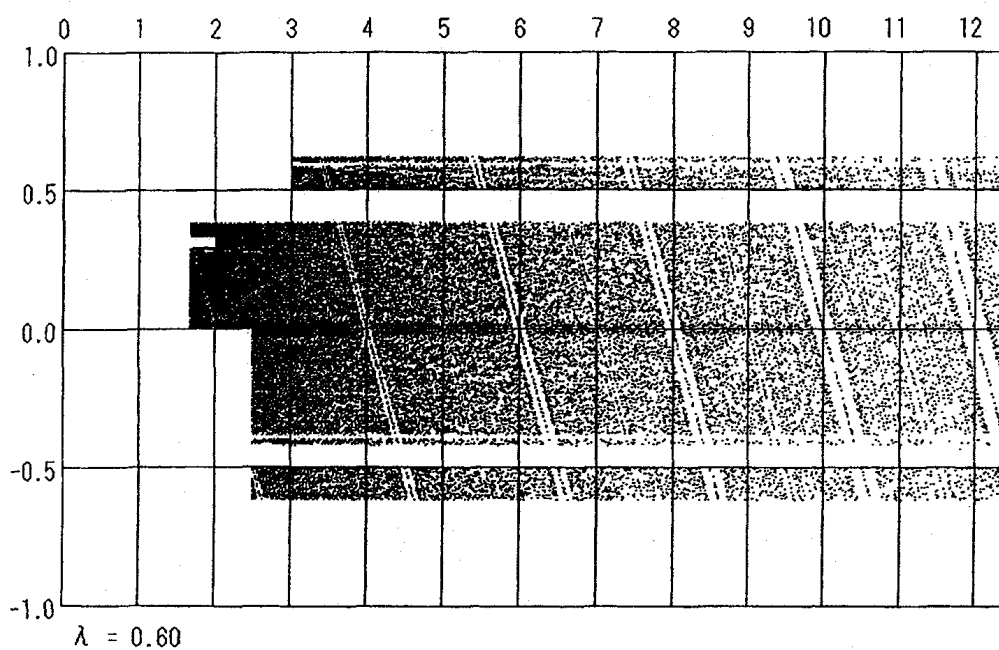
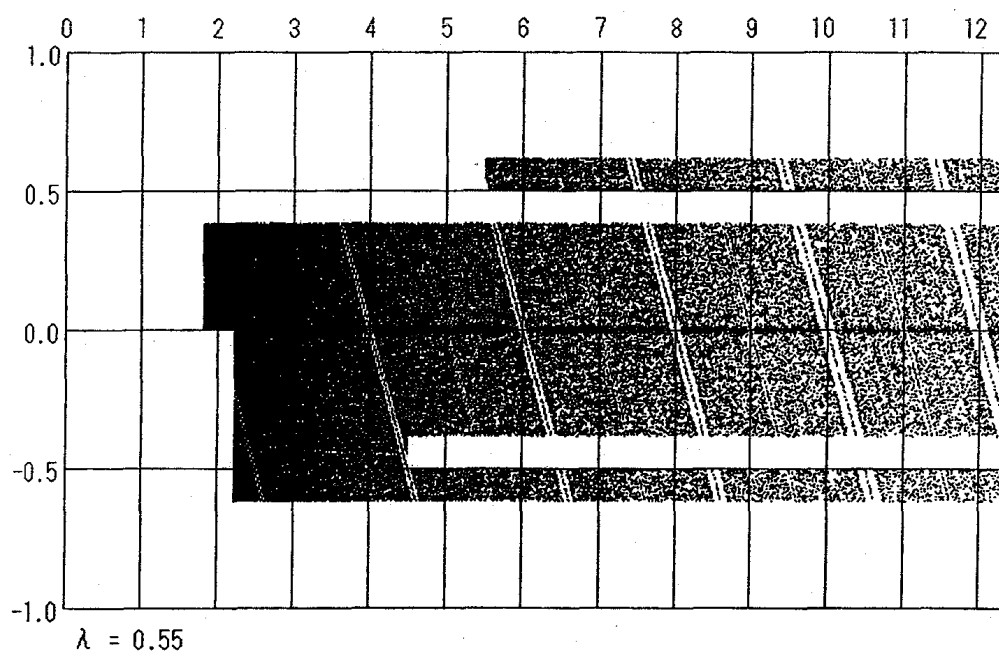
5. Reduced sets for $\lambda > 1/2$. In the previous section we gave the reduced sets R_λ for $0 \leq \lambda \leq 1/2$. Moreover we know the set for R_1 . What are R_λ 's for $1/2 < \lambda < 1$? Our experiments have shown that they seem to be very complicated (See Appendix).

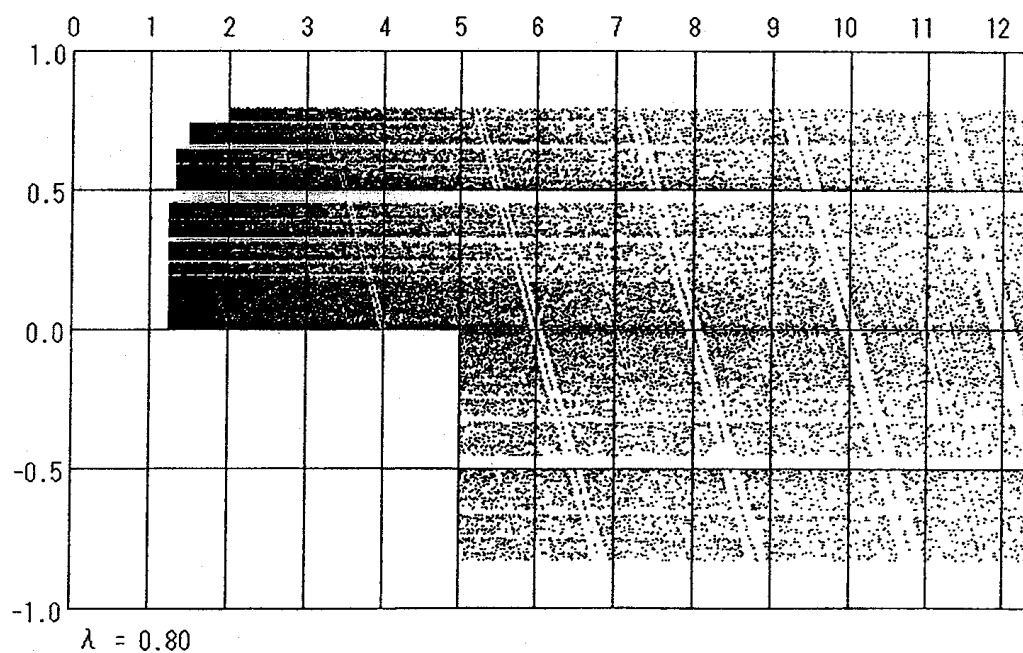
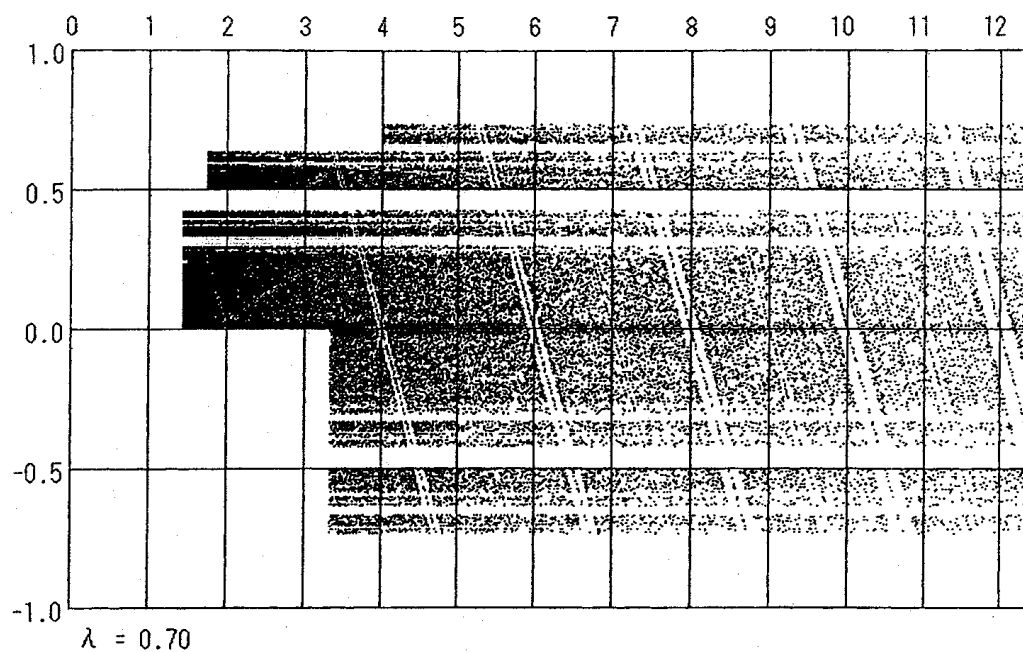
Appendix. For a positive non-square integer D we set $\omega = (1 + \sqrt{D})/2$ or \sqrt{D} according to $D \equiv 1 \pmod{4}$ or not, and we compute $\omega_k = \rho_\lambda^k(\omega)$ for $k = 0, 1, 2, \dots$. Then we know that $\omega_1 = \rho_\lambda^n(\omega)$ for some $n > 1$. Thus we plot all points $(\omega_k, \overline{\omega_k})$ for $k = 1, 2, \dots, n-1$ while D runs from 2 through 5000. The following figures show the results obtained in such a way for various λ 's.

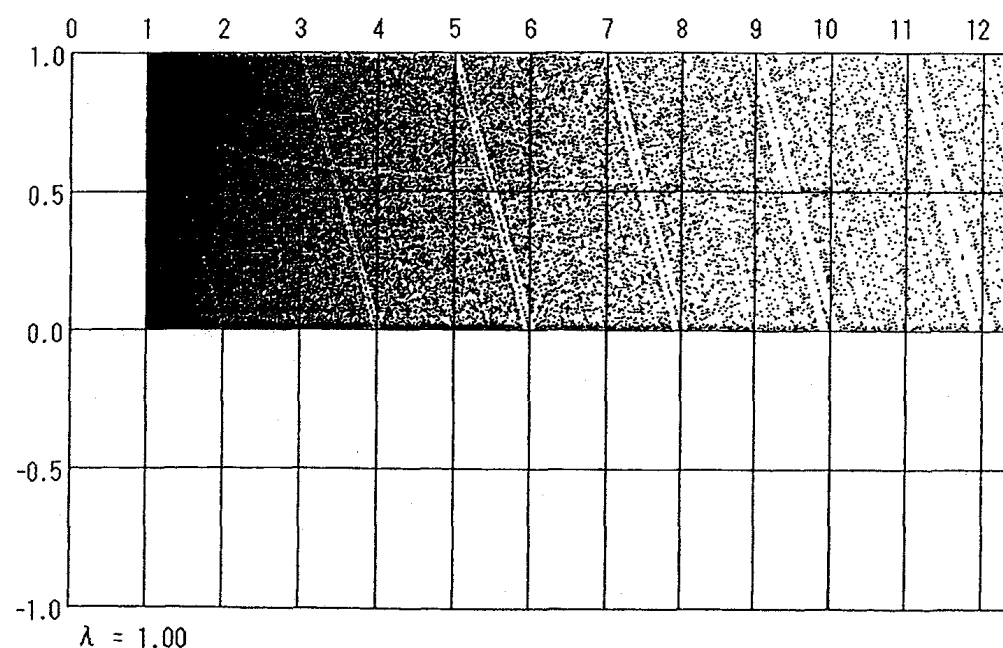
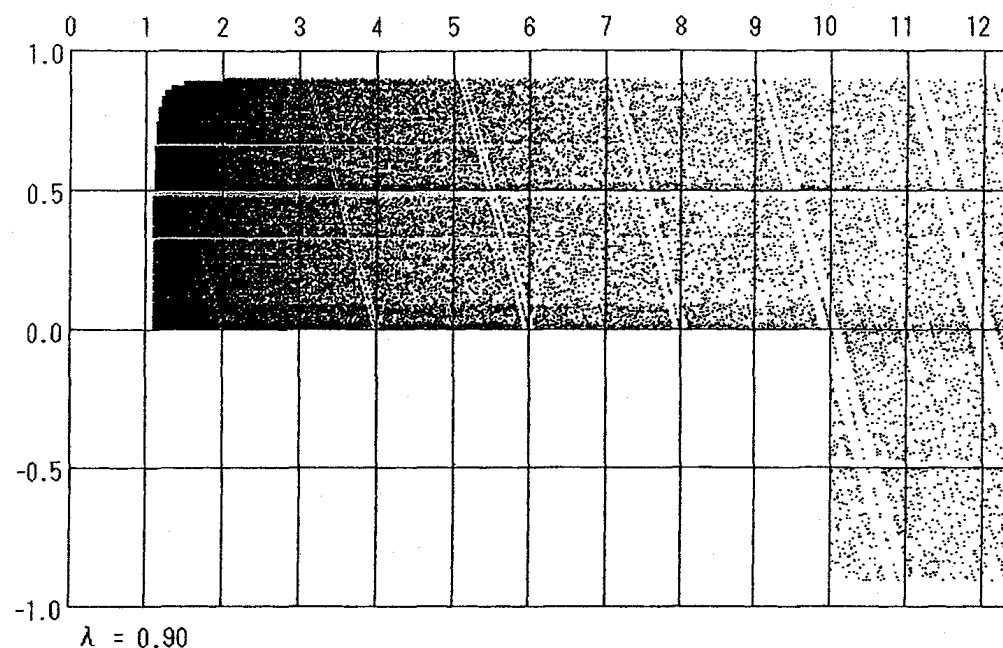












REFERENCES

1. Y. Mimura, *On odd solutions of the equation $X^2 - DY^2 = 4$* , Proceedings of the symposium on analytic number theory and related topics, ed. By K. Nagasaka, World Scientific Co. (1993), 87-96.
2. P. Kaplan and Y. Mimura, *Développement en fraction continue à l'entier le plus proche, idéaux α -réduits et un problème d'Eisenstein*, Acta Arithmetica **76** (1996), 285-304.
3. —, *Développement en fraction continue à l'entier supérieur, idéaux 0-réduits et un problème d'Eisenstein*, Acta Arithmetica **78** (1997), 275-285.