《Article》

N2- AND N3-LATTICES OVER REAL QUADRATIC INTEGERS

Yoshio Mimura
(Kobe Pharmaceutical University)

Abstract
Positive definite integral quadratic lattices(or the associated quadratic forms) have been investigated from various points of views, one of which is a system of generators with some specified lengths. In this paper we will find all lattices(over real quadratic integers) which are generated by some vectors the norms of lengths of which are 2 or 3.

0. Introduction

Recall 2-lattices. Let F be a totally real algebraic field and o be the ring of integers in F . Let V be a totally positive definite quadratic space over F with symmetric bilinear form B and the associated quadratic form Q. A vector x in V is called a q-vector if $\mathrm{Q}(x) = q$. A lattice L, a finitely generated o-module in V, is said to be integral if $\mathrm{B}(L, L) \subset o$. It is clear that $\mathrm{Q}(L) \subset o$ if L is integral. An integral lattice L in V is called a q-lattice if it is generated by some q-vectors over o. Note that any 1-lattce is isometric to a unit lattice whose matrix is the identity

*Received October 31, 2001. Accepted December 22, 2001.

matrix. Every 2-lattice is even. When F is the rational number field $\mathbb{Q}$, every indecomposable 2-lattice falls in one of the three infinite sequences or the exceptional set consisting of three lattices E_6, E_7 and E_8 [1,2]. It is well known that Leech lattice is a 4-lattice. When F is a real quadratic field, every indecomposable 2-lattices falls in one of four infinite sequences or the exceptional set consisting of five lattices [2]. Also see [3] for 2-lattices in an hermitian space over an imaginary quadratic field. In this case there are 2-lattices which is not a free o-module.

From now on we suppose that $q = 2, 3$ and $F = \mathbb{Q}(\sqrt{m})$ is a real quadratic field with a square-free integer $m > 1$. We denote the conjugate of a number $\alpha \in F$ by $\overline{\alpha}$, and define the norm of α by $\mathrm{N}(\alpha) = \alpha\overline{\alpha}$ and the trace of α by $\mathrm{tr}(\alpha) = \alpha + \overline{\alpha}$. Note that $\mathrm{N}(\alpha)$ and $\mathrm{tr}(\alpha)$ are rational integers for any $\alpha \in o$. A number $\alpha \in F$ is said to be totally positive, and written as $\alpha \succ 0$, if $\alpha > 0$ and $\overline{\alpha} > 0$. Let u be the unit group of o, and put

$$F^+ = \{\alpha \in F \mid \alpha \succ 0\}, o^+ = o \cap F^+, u^+ = u \cap F^+.$$

Note that $\alpha\beta$, $\alpha + \beta \in F^+$ if $\alpha, \beta \in F^+$. Let ε_0 be the fundamental unit > 1. Then u^+ is generated by the unit ε_1, where $\varepsilon_1 = \varepsilon_0$ or $\varepsilon_1 = \varepsilon_0^2$ according as $\mathrm{N}(\varepsilon_0)$=1 or -1. A totally positive integer α is said to be irreducible if α cannot be expressed as $\alpha = \alpha_1 + \alpha_2$ with $\alpha_1, \alpha_2 \in o^+$

Lemma 0. 1. *Let $\alpha \in o^+$. Then $\mathrm{tr}(\alpha) \geq 2$, and the equality holds if and only if $\alpha = 1$. And if $\mathrm{N}(\alpha) \leq 3$, then α is irreducible.*

Proof. If $\mathrm{tr}(\alpha) = 1$, then we have $0 < \alpha = (1 + k\sqrt{m}\,) / 2$ with some rational odd integer k and $m \equiv 1 \pmod 4$. This is a contradiction. If $\mathrm{tr}(\alpha) = 2$, then we have $0 < \alpha = 1 + k\sqrt{m}$ with some rational integer k. Thus $k = 0$ or $\alpha = 1$. Suppose that $\alpha = \alpha_1 + \alpha_2$ with $\alpha_1, \alpha_2 \in o^+$.

Then we have

$$3 \geq \mathrm{N}(\alpha) = \mathrm{N}(\alpha_1) + \mathrm{N}(\alpha_2) + \mathrm{tr}(\alpha_1\overline{\alpha_2}) \geq 1 + 1 + 2 = 4,$$

which is a contradiction. □

By M_q we denote the set of square-free rational integers $m\,(>1)$ for which there is a totally positive integer α such that $\mathrm{N}(\alpha) = q$ in $F = \mathbb{Q}(\sqrt{m})$. Then the ideal(q) can be written as $(q) = \boldsymbol{q}\overline{\boldsymbol{q}}$ with $\boldsymbol{q} = (\alpha)$ in F. We have two cases: $\boldsymbol{q} = \overline{\boldsymbol{q}}$ and $\boldsymbol{q} \neq \overline{\boldsymbol{q}}$. We define the subset M_q^1 (resp. M_q^2) of M_q for which $\boldsymbol{q} = \overline{\boldsymbol{q}}$ (resp. $\boldsymbol{q} \neq \overline{\boldsymbol{q}}$). Clearly the following statements hold: If $m \in M_2^1$, then $m \equiv 2, 3 \pmod 4$. If $m \in M_2^2$, then $m \equiv 1 \pmod 8$. If $m \in M_3^1$, then $m \equiv 0 \pmod 3$. If $m \in M_3^2$, then $m \equiv 1 \pmod 3$. For $m \in M_q$ we put

$$\pi_q = \min\{\, \alpha \in o^+ \mid \mathrm{N}(\alpha) = q,\ \alpha \geq \overline{\alpha} \,\}$$

Remark 0. 1. *By a simple computation we have the following:*

$$M_2^1 = \{2, 7, 14, 23, 31, 34, 46, 47, 62, 71, 79, 94, 103, 119, 127, \ldots\}$$

$$M_2^2 = \{17, 41, 73, 89, 97, 113, 137, 161, 193, 217, 233, 241, 281, \ldots\}$$

$$M_3^1 = \{6, 33, 69, 78, 141, 177, 213, 222, 249, 321, 366, 393, 429, \ldots\}$$

$$M_3^2 = \{13, 22, 37, 46, 61, 73, 94, 97, 109, 118, 157, 166, 181, 193, \ldots\}$$

Lemma 0. 2. *If $\alpha \in o^+$ with $\mathrm{N}(\alpha) = q$, then $\alpha = \varepsilon_1^r \pi_q$ or $\alpha = \varepsilon_1^r \overline{\pi}_q$ with $r \in \mathbb{Z}$. If $m \in M_q^1$, then $\pi_q = \overline{\pi}_q \varepsilon_1$. Moreover, if $m \in M_q^1$ and $m > 2$ then $\varepsilon_1 = \varepsilon_0$.*

Proof. Clearly we can write $\alpha = \varepsilon_1^r \pi_q$ or $\alpha = \varepsilon_1^r \overline{\pi}_q$ with $r \in \mathbb{Z}$. If $m \in M_q^1$, then we see that $\pi_q / \overline{\pi}_q = \varepsilon_1^r$ for some positive rational integer r. If $r>1$,we put $\gamma = \pi_q \varepsilon_1^{-1}$ and see that $\gamma - \overline{\gamma} = \overline{\pi}_q \varepsilon_1 (\varepsilon_1^{r-2} - 1) \geq 0$. This contradicts the minimality of π_q. If $\varepsilon_1 = \varepsilon_0^2$, then we have $\pi_q = \overline{\pi}_q \varepsilon_0^2$. Thus we see that $\pi_q^2 = q\varepsilon_0^2$, which implies $\sqrt{q} \in F$ or $m = q$. By Remark 0.1 we have $m = q = 2$. □

1. Nq-lattices

Let q be a positive rational integer. A vector $x \in V$ is called an Nq-vector if $\mathrm{N}(\mathrm{Q}(x)) = q$.Let L be an integral lattice L in a totally positive definite quadratic space V over F. L is called an Nq-lattice if L is generated by a finite number of Nq-vectors. In this paper we will treat Nq-lattices for $q = 2, 3$.

We say that a lattice L is indecomposable if there are no sublattices L_1 and L_2 so that $L = L_1 \perp L_2$ (that is, $L = L_1 + L_2$ as o-modules and $\mathrm{B}(L_1, L_2)=\{0\}$). Clearly every lattice is the orthogonal sum of some indecomposable sublattices.

Lemma1. 1. *Every lattice L has the unique orthogonal splitting $L = L_1 \perp L_2 \perp \cdots \perp L_t$ (but for their oder).*

See [4, Theorem 105:1].

Proposition1. 1. *Let* $q = 1, 2, 3$, *and assume that an integral lattice L is the orthogonal sum of two sublattices L_1 and L_2. Then L is an* Nq*-lattice if and only if L_i is an* Nq*-lattice for $i = 1, 2$.*

Proof. Clearly L is an Nq-lattice if L_1 *and* L_2 are Nq-lattices. So we shall show that L_1 is an Nq-lattce if L is so. Let $x \in L$ be an Nq-vector, and suppose $x = y + z$ with $y \in L_1$ and $z \in L_2$ .Then we have $\mathrm{Q}(x)=\mathrm{Q}(y)+\mathrm{Q}(z)$, which implies that $\mathrm{Q}(y) = 0$ or $\mathrm{Q}(z) = 0$ from Lemma 0. 1. Thus we have $y = 0$ or $z = 0$ because Q is a totally positive quadratic form. Hence we can assume that L has a system of generators $\{x_1, \ldots, x_r, x_{r+1}, \ldots, x_m\}$ with $x_1, \ldots, x_r \in L_1$ and $x_{r+1}, \ldots, x_m \in L_2$. Now take a vector $x \in L_1$. Then we can write $x = \Sigma_{i \leq r}\alpha_i x_i + \Sigma_{j > r}\alpha_j x_j$ with $\alpha_k \in o$, $1 \leq k \leq m$. Hence we can see that $x - \Sigma_{i \leq r}\alpha_i x_i = \Sigma_{j > r}\alpha_j x_j \in L_1 \cap L_2 = \{0\}$, or that $x = \Sigma_{i \leq r}\alpha_i x_i$. Therefore L_1 is an Nq-lattice. □

By Proposition 1. 1 we have only to find all indecomposable Nq-lattices. If a lattice L is a free o-module, then we can write $L = ox_1 + ox_2 + \cdots + ox_n$ with linearly independent vectors x_i 's in V. Then we define the discriminant disc(L) of L by the determinant of the matrix the (i, j)-entry of which is $B(x_i, x_j)$. This value is independent of the choice of x_i 's up to squares of units. We say that an integral lattice L is unimodular if disc $(L) \in u$.

Lemma 1. 2. *If M is a unimodular sublattice of an integral lattice L, then M splits L.*

See [4, Proposition 82: 15].

Let $x_1, ..., x_m \in V$. By $<x_1, \ldots, x_m>$ we mean the o-module generated by $x_1, \ldots, x_m$. When $L = <x_1, \ldots, x_m>$ we write $L \cong A$, where A is the matrix the (i, j)-entry of which is $B(x_i, x_j)$. Note that det $(A) > 0$ or det $(A) = 0$. The latter case occurs only when $x_1, \ldots, x_m$ are linearly dependent over F. Suppose that L is an Nq-lattice generated by Nq-vectors $x_1, \ldots, x_n$. We say that $\{x_1, \ldots, x_n\}$ is a minimal Nq-system for L if L has no Nq-system which consists of $n - 1$ vectors. Here we note that vectors $x_1, \ldots, x_n$ are not always linearly independent.

Proposition 1. 2. *Assume that L is an indecomposable Nq-lattice with a minimal Nq-system $\{x_1, \ldots, x_n\}$. Then there is a permutation p of $\{x_1, \ldots, n\}$ so that the sublattices $<x_{p(1)}, \ldots, x_{p(r)}>$ are indecomposable Nq-lattices with minimal Nq-systems $\{x_{p(1)}, \ldots, x_{p(r)}\}$ for $r = 1, \ldots, n$.*

Proof. Consider a graph G the vertex set of which is $\{x_1, \ldots, x_n\}$ where two vertices x_i and x_j are adjacent if and only if $B(x_i, x_j) \neq 0$. Then G is connected since L is indecomposable. By an easy argument in graph theory, we can eliminate a vertex x_k from G so that the subgraph $G - \{x_k\}$ is connected, which means that the sublattice $<x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n>$ is indecomposable since x_i 's are not the sum of two nonzero vectors in L. □

Remark 1. 1. *Let L be an indecomposable $N1$-lattice. Then L is isometric to*

[1] *if* $\varepsilon_1 = \varepsilon_0^2$. *On the other hand, L is isometric to* [1] *or* $[\varepsilon_0]$ *if* $\varepsilon_1 = \varepsilon_0$.

2. N2-lattices

Let L be an N2-lattice with a minimal N2-system $\{x_1, \ldots, x_n\}$. Write $\pi = \pi_2$ and $\boldsymbol{q} = (\pi)$. We can easily see the following proposition.

Proposition 2. 1. *Suppose $n = 1$. Then L is isometric to $J_{1,0}$, $\overline{J}_{1,0}$, $J_{1,1}$, or $\overline{J}_{1,1}$:*

$$J_{1,0} \cong [\pi],\ \overline{J}_{1,0} \cong [\overline{\pi}],\ J_{1,1} \cong [\varepsilon_1 \pi],\ \overline{J}_{1,1} \cong [\overline{\varepsilon_1 \pi}]\ .$$

All isomtric relations are:

$$J_{1,0} \cong \overline{J}_{1,0} \Longleftrightarrow J_{1,1} \cong \overline{J}_{1,1} \Longleftrightarrow m = 2,$$
$$J_{1,0} \cong J_{1,1} \Longleftrightarrow \overline{J}_{1,0} \cong \overline{J}_{1,1} \Longleftrightarrow \varepsilon_1 = \varepsilon_0^2,$$
$$J_{1,0} \cong \overline{J}_{1,1} \Longleftrightarrow \overline{J}_{1,0} \cong J_{1,1} \Longleftrightarrow m \in M_2^1.$$

Now suppose $n = 2$. Then we have

$$L = \langle x_1, x_2 \rangle \cong \begin{bmatrix} \alpha & \lambda \\ \lambda & \beta \end{bmatrix},$$

where $\alpha, \beta \in o^+$ with $N(\alpha) = N(\beta) = 2$ and $0 \neq \lambda \in o$. Put $\delta = \alpha\beta - \lambda^2$. Then $\delta > 0$ or $\delta = 0$. If $\delta = 0$, then we have $(\alpha) = (\beta) = (\lambda) = \boldsymbol{q}$ or $\overline{\boldsymbol{q}}$ and $Q(\alpha x_2 - \lambda x_1) = 0$ (that is, $\alpha x_2 - \lambda x_1 = 0$). Hence $L = ox_1 + ox_2 = ox_1 + \lambda\alpha^{-1}ox_1 = ox_1$. This is a contradiction to the minimality of the N2-system $\{x_1, x_2\}$. Thus $\delta > 0$. We need the following lemma.

Lemma 2. 1. *Let* $\gamma = \gamma_1 + \gamma_2$ *with* $\gamma_1, \gamma_2 \in o^+$. *If* $N(\gamma) = 4$ *then* $\gamma_1 = \gamma_2 \in u^+$.

Proof We have $4 = N(\gamma_1) + N(\gamma_2) + \mathrm{tr}(\gamma_1 \overline{\gamma_2})$ with $N(\gamma_1) \geq 1$, $N(\gamma_2) \geq 1$ and $\mathrm{tr}(\gamma_1 \overline{\gamma_2}) \geq 2$. Hence all the equalities and $\gamma_1 \overline{\gamma_2} = 1$ should hold. □

Applying Lemma 2. 1 to $\alpha\beta = \lambda^2 + \delta$ we see that $\delta = \lambda^2 \in u^+$, and $\alpha\beta = 2\lambda^2$. Replacing x^2 by $\overline{\lambda} x_2$, we have $\beta = \overline{\alpha}$, $\lambda = 1$, and $\delta = 1$. We can assume $\alpha = \pi$ or $\alpha =$

$\varepsilon_1\pi$ by using a minimal N2-system $\{\eta x_1, \bar{\eta} x_2\}$ or $\{\eta x_2, \bar{\eta} x_1\}$ with some $\eta \in u$ if necessary. By Lemma 0.2 we have

Proposition 2. 2. *Suppose $n = 2$ and $m \in M_2$. Then L is isometric to $J_{2,0}$ or $J_{2,1}$:*

$$J_{2,0} \cong \begin{bmatrix} \pi & 1 \\ 1 & \bar{\pi} \end{bmatrix}, \qquad J_{2,1} \cong \begin{bmatrix} \varepsilon_1\pi & 1 \\ 1 & \overline{\varepsilon_1\pi} \end{bmatrix},$$

where $\mathrm{disc}(J_{2,0}) = \mathrm{disc}(J_{2,1}) = 1$. *And $J_{2,0} \cong J_{2,1}$ if and only if $m \in M_2^1$ or $\varepsilon_1 = \varepsilon_0^2$.*

Proof. We have only to show that $J_{2,0}$ does not represent $\varepsilon_1\pi$ if $m \in M_2^2$ and $\mathrm{N}(\varepsilon_0) = 1$. If $\pi\xi^2 + 2\xi\eta + \bar{\pi}\eta^2 = \varepsilon_1\pi$ for some $\xi, \eta \in o$, then we have $\xi \neq 0, \eta \neq 0$ and $\xi^2 + (\xi + \bar{\pi}\eta)^2 = 2\varepsilon_1$. Hence, by Lemma 2.1, we see that $\xi^2 = \varepsilon_1$, which is a contradiction. □

Theorem 2. 1. *All indecomposable* N2*-lattices are given in Propositions 2.1 and 2.2.*

Proof This is a consequence from Lemma 1.2, Proposition 1.2 and the fact that the lattices in Proposition 2.2 are unimodular. □

3. N3-lattices

Let L be an N3-lattice with a minimal N3-system $\{x_1, \ldots, x_n\}$. Write $\pi = \pi_3$ and $\boldsymbol{q} = (\pi)$. We can easily see the following proposition.

Proposition 3.1. *Suppose $n = 1$. Then L is isometric to $K_{1,0}$, $\overline{K}_{1,0}$, $K_{1,1}$, or $\overline{K}_{1,1}$:*

$$K_{1,0} \cong [\pi],\ \overline{K}_{1,0} \cong [\bar{\pi}],\ K_{1,1} \cong [\varepsilon_1\pi],\ \text{or}\ \overline{K}_{1,1} \cong [\overline{\varepsilon_1\pi}],$$

and $K_{1,0} \cong \overline{K}_{1,0}$, $K_{1,1} \cong \overline{K}_{1,1}$. *All isometric relation are:*

$$K_{1,0} \cong K_{1,1} \Longleftrightarrow \overline{K}_{1,0} \cong \overline{K}_{1,1} \Longleftrightarrow \varepsilon_1 = \varepsilon_0^2,$$

$$K_{1,0} \cong \overline{K}_{1,1} \Longleftrightarrow \overline{K}_{1,0} \cong K_{1,1} \Longleftrightarrow m \in M_3^1,$$

Now suppose $n = 2$. Then we have

$$L = \langle x_1, x_2 \rangle \cong \begin{bmatrix} \alpha & \lambda \\ \lambda & \beta \end{bmatrix},$$

where $\alpha, \beta \in o^+$ with $\mathrm{N}(\alpha) = \mathrm{N}(\beta) = 3$ and $0 \neq \lambda \in o$. Put $\delta = \alpha\beta - \lambda^2$. Then $\delta > 0$ or $\delta = 0$. If $\delta = 0$, then we have $(\alpha) = (\beta) = (\lambda) = \boldsymbol{q}$ or $\overline{\boldsymbol{q}}$ and $\mathrm{Q}(\alpha x_2 - \lambda x_1) = 0$ (that is, $\alpha x_2 - \lambda x_1 = 0$). Hence $L = ox_1 + ox_2 = ox_1 + \lambda\alpha^{-1}ox_1 = ox_1$. This is a contradiction to the minimality of the N3-system $\{x_1, x_2\}$. Thus $\delta > 0$. We need the following lemma.

Lemma 3. 1. *Let $m \in M_3$ and $\gamma = \gamma_1 + \gamma_2$ with $\gamma_1, \gamma_2 \in o^+$. If* $\mathrm{N}(\gamma) = 9$ *and* $\mathrm{N}(\gamma_1) \geq \mathrm{N}(\gamma_2)$, *then $\gamma_2 \in u$ and one of the following holds: (1)$m = 13$, $\gamma_1 = \pi\gamma_2$ or $\overline{\pi}\gamma_2$, (2)$\gamma_1 = 2\gamma_2$.*

Proof. We have $9 = \mathrm{N}(\gamma_1) + \mathrm{N}(\gamma_2) + \mathrm{tr}(\gamma_1\overline{\gamma_2})$ with $\mathrm{N}(\gamma_1) \geq 1$, $\mathrm{N}(\gamma_2) \geq 1$, and $t = \mathrm{tr}(\gamma_1\overline{\gamma_2}) \geq 2$. Hence $\gamma_1\overline{\gamma_2} = (t + k\sqrt{m})/2$ for some rational integer k with $t \equiv k \pmod 4$, so $\mathrm{N}(\gamma_1)\,\mathrm{N}(\gamma_2) = (t^2 - k^2 m)/4$. Among all possible values of $\mathrm{N}(\gamma_1)$, $\mathrm{N}(\gamma_2)$ and t we have two cases (note that $5, 7 \notin M_3$): (i) $\mathrm{N}(\gamma_1) = 3$, $\mathrm{N}(\gamma_2) = 1$, $\gamma_1\overline{\gamma_2} = (5 \pm \sqrt{13})/2$ and (ii) $\mathrm{N}(\gamma_1) = 2$, $\mathrm{N}(\gamma_2) = 1$, $\gamma_1\overline{\gamma_2} = 2$. □

Applying Lemma 3.1 to the case $\alpha\beta = \lambda^2 + \delta$ with $\lambda^2, \delta \in o^+$, we have the following three cases:

(*CK*) $m = 13$. $\lambda \in u^+$, $\delta = \pi\lambda^2$ or $\overline{\pi}\lambda^2$, $\alpha\beta = \varepsilon_1\overline{\pi}^2\lambda^2$ or $\overline{\varepsilon_1}\pi^2\lambda^2$.

(*CM*) $\lambda \in u^+$, $\delta = 2\lambda^2$, $\beta = \overline{\alpha}\lambda^2$.

(*CP*) $\delta \in u^+$, $\lambda^2 = 2\delta$, $\beta = \overline{\alpha}\delta$.

First consider the case (*CK*). Note that $\pi + 1 = \varepsilon_1\overline{\pi}^2$ and $(\alpha) = (\beta)$. Since $\varepsilon_1 = \varepsilon_0^2$, we may assume that $\alpha = \pi$ or $\overline{\pi}$. If $\alpha = \pi$, then $\beta = \overline{\varepsilon_1}\pi\lambda^2$ and $\delta = \overline{\pi}\lambda^2$. Hence $L = \langle x_1, x_2 \rangle = \langle x_1, \varepsilon_0\lambda^{-1}x_2 \rangle \cong K_2$ in Proposition 3.2. Similarly we have $\overline{K}_2$ when $\alpha = \overline{\pi}$. A simple computation shows that K_2 does not represent $\overline{\pi}$, so $K_2 \not\cong \overline{K}_2$. In this case we say that L is of type *TK*.

Secondly consider the case (*CM*). Note that $(\alpha) = (\overline{\beta})$. It can be easily seen that L is isometric to $M_{2,0}$ or $M_{2,1}$ in Proposition 3.2. A simple computation shows that $M_{2,0} \cong M_{2,1}$ if and only if $m \in M_3^1$ or $\varepsilon_1 = \varepsilon_0^2$. In this case we say that L is of type TM.

Finally consider the case (CP). We may assume $\delta = \varepsilon_1$ since $m \neq 2$. Fix a ρ so that $\rho^2 = 2\varepsilon_1$. Then we see that L is isometric to P_2 in Proposition 3.2 or $\overline{P}_2$ (the conjugate of P_2). And we see

$$P_2 = <x_1, x_2> = <\overline{\pi}x_1 - \rho\overline{\varepsilon_1}x_2, \pm \overline{\varepsilon_1}\,(\rho x_1 - \pi x_2)> \cong \overline{P}_2.$$

Remark 3. 1. *By* M_3' *we denote the set of* $m \in M_3$ *so that* $\rho^2 = 2\varepsilon_1$ *for some* $\rho \in \mathbb{Q}(\sqrt{m})$. *If* $m \in M_3'$ *then* $\varepsilon_1 = \varepsilon_0$ *and* $m \equiv 6, 10 (\mathrm{mod}\ 12)$. *We see*

$$M_3' = \{6, 22, 46, 94, 118, 166, 214, 262, 334, \ldots\}.$$

Proposition 3. 2. *Suppose* $n = 2$ *and* $m \in M_3$. *Then* L *is isometric to one of the following:*

$$K_2 \cong \begin{bmatrix} \pi & \varepsilon_0 \\ \varepsilon_0 & \pi \end{bmatrix}, \quad \overline{K}_2 \cong \begin{bmatrix} \overline{\pi} & \overline{\varepsilon_0} \\ \overline{\varepsilon_0} & \overline{\pi} \end{bmatrix}, \quad P_2 \cong \begin{bmatrix} \pi & \rho \\ \rho & \varepsilon_1\overline{\pi} \end{bmatrix}$$

$$M_{2,0} \cong \begin{bmatrix} \pi & 1 \\ 1 & \overline{\pi} \end{bmatrix}, \quad M_{2,1} \cong \begin{bmatrix} \varepsilon_1\pi & 1 \\ 1 & \overline{\varepsilon_1\pi} \end{bmatrix},$$

where $\rho \in o$ *so that* $\rho^2 = 2\varepsilon_1$. K_2 *and* $\overline{K}_2$ *appear when* $m = 13$. P_2 *appears when* $m \in M_3'$. $\mathrm{disc}(K_2) = \varepsilon_1\overline{\pi}$, $\mathrm{disc}(\overline{K}_2) = \overline{\varepsilon_1}\pi$, $\mathrm{disc}(P_2) = \varepsilon_1$, $\mathrm{disc}(M_{2,0}) = \mathrm{disc}(M_{2,1}) = 2$. $K_2 \ncong \overline{K}_2$, $M_{2,0} \cong M_{2,1}$ *if and only if* $m \in M_3^1$ *or* $\varepsilon_1 = \varepsilon_0^2$.

Let $n = 3$ and $\{x_1, x_2, x_3\}$ be a minimal N3-system for an indecomposable N3-lattice L. Then the lattices $<x_i, x_j>$ are of type TK, TM or TD for $1 \leq i \leq j \leq 3$, where we say that a binary decomposable N3-lattice is of type TD. A case $\{T_{12}, T_{13}, T_{23}\}$ means that $<x_i, x_j>$ is of type T_{ij} for $1 \leq i \leq j \leq 3$. Clearly the cases $\{TM, TM, TM\}$ and $\{TK, TK, TM\}$ do not appear.

1) Case $\{TK, TK, TK\}$. From (CK) we can assume that L is isometric to

$$K_3 \cong \begin{bmatrix} \pi & \varepsilon_0 & \varepsilon_0 \\ \varepsilon_0 & \pi & \eta \\ \varepsilon_0 & \eta & \pi \end{bmatrix} \quad \text{or} \quad \overline{K}_3 \cong \begin{bmatrix} \overline{\pi} & \overline{\varepsilon_0} & \overline{\varepsilon_0} \\ \overline{\varepsilon_0} & \overline{\pi} & \overline{\eta} \\ \overline{\varepsilon_0} & \overline{\eta} & \overline{\pi} \end{bmatrix}$$

where $\eta = \pm\varepsilon_0$. Since $\mathrm{disc}(K_3) = \varepsilon_1\ (1 + 2\ (\eta - \varepsilon_0))$, we see that $\eta = \varepsilon_0$. We see that

$$\overline{K_3} = <x_1, x_2, x_3> = <\pi x_1 + \varepsilon_0 x_2 + \varepsilon_0 x_3, \varepsilon_0 x_1 + \pi x_2 + \varepsilon_0 x_3, \varepsilon_0 x_1 + \varepsilon_0 x_2 + \pi x_3> \cong K_3.$$

2) Case $\{TM, TM, TM, TK\}$. From (CM) and (CK) we can assume that L is isometric to

$$A \cong \begin{bmatrix} \overline{\pi} & 1 & 1 \\ 1 & \pi & \eta \\ 1 & \eta & \pi \end{bmatrix} \quad \text{or} \quad \overline{A} \cong \begin{bmatrix} \pi & 1 & 1 \\ 1 & \overline{\pi} & \overline{\eta} \\ 1 & \overline{\eta} & \overline{\pi} \end{bmatrix},$$

where $\eta = \pm\varepsilon_0$. Then $\mathrm{disc}\ (A) = -\varepsilon_0 + 2\eta$, which is not totally positive nor 0. Thus this case does not occur.

3) Case $\{TK, TK, TD\}$. From(CK) we can assume that L is isometric to

$$A \cong \begin{bmatrix} \pi & \varepsilon_0 & \varepsilon_0 \\ \varepsilon_0 & \pi & 0 \\ \varepsilon_0 & 0 & \pi \end{bmatrix} \quad \text{or} \quad \overline{A} \cong \begin{bmatrix} \overline{\pi} & \overline{\varepsilon_0} & \overline{\varepsilon_0} \\ \overline{\varepsilon_0} & \overline{\pi} & 0 \\ \overline{\varepsilon_0} & 0 & \overline{\pi} \end{bmatrix},$$

and their discriminants are $-\pi\varepsilon_0$ or $-\overline{\pi\varepsilon_0}$, which are not totally positive nor 0.

4) Case $\{TM, TM, TD\}$. From (CM) we can assume that L is isometric to

$$M_{3,0} \cong \begin{bmatrix} \pi & 1 & 1 \\ 1 & \overline{\pi} & 0 \\ 1 & 0 & \overline{\pi} \end{bmatrix}, \quad \overline{M}_{3,0} \cong \begin{bmatrix} \overline{\pi} & 1 & 1 \\ 1 & \pi & 0 \\ 1 & 0 & \pi \end{bmatrix},$$

$$M_{3,1} \cong \begin{bmatrix} \pi\varepsilon_1 & 1 & 1 \\ 1 & \overline{\pi\varepsilon_1} & 0 \\ 1 & 0 & \overline{\pi\varepsilon_1} \end{bmatrix}, \quad \overline{M}_{3,1} \cong \begin{bmatrix} \overline{\pi\varepsilon_1} & 1 & 1 \\ 1 & \pi\varepsilon_1 & 0 \\ 1 & 0 & \pi\varepsilon_1 \end{bmatrix},$$

where $\mathrm{disc}\ (M_{3,0}) = \overline{\pi}$, $\mathrm{disc}\ (\overline{M}_{3,0}) = \pi$, $\mathrm{disc}\ (M_{3,1}) = \overline{\pi\varepsilon_1}$, $\mathrm{disc}\ (\overline{M}_{3,1}) = \pi\varepsilon_1$. If $m \in M_3^2$ and $\varepsilon_1 = \varepsilon_0$, we see that any two of them are not isometric by considering the discriminants. It can be easily seen that $M_{3,0} \cong M_{3,1}$ and $\overline{M}_{3,0} \cong \overline{M}_{3,1}$ if $m \in M_3^2$ with $\varepsilon_1 = \varepsilon_0^2$ and that $M_{3,0} \cong \overline{M}_{3,1}$ and $\overline{M}_{3,0} \cong M_{3,1}$ if $m \in M_3^1$.

5) Case $\{TK, TM, TD\}$. From (CK) and (CM) we can assume that L is isometric

to

$$A \cong \begin{bmatrix} \pi & \varepsilon_0 & 1 \\ \varepsilon_0 & \pi & 0 \\ 1 & 0 & \overline{\pi} \end{bmatrix}, \quad \overline{A} \cong \begin{bmatrix} \overline{\pi} & \overline{\varepsilon_0} & 1 \\ \overline{\varepsilon_0} & \overline{\pi} & 0 \\ 1 & 0 & \pi \end{bmatrix},$$

We note that $A = \langle x_1, x_2, x_3 \rangle = \langle x_1, x_2, \varepsilon_0 x_1 - x_2 - \varepsilon_0 x_3 \rangle \cong K_3$. Similarly $\overline{A} \cong \overline{K}_3 \cong K_3$.

From the above observation we have the following proposition:

Proposition 3. 3. *Suppose $n = 3$ and $m \in M_3$. Then L is isometric to one of the following:*

$$K_3 \cong \begin{bmatrix} \pi & \varepsilon_0 & \varepsilon_0 \\ \varepsilon_0 & \pi & \varepsilon_0 \\ \varepsilon_0 & \varepsilon_0 & \pi \end{bmatrix}, \quad M_{3,0} \cong \begin{bmatrix} \pi & 1 & 1 \\ 1 & \overline{\pi} & 0 \\ 1 & 0 & \overline{\pi} \end{bmatrix}, \quad \overline{M}_{3,0} \cong \begin{bmatrix} \overline{\pi} & 1 & 1 \\ 1 & \pi & 0 \\ 1 & 0 & \pi \end{bmatrix},$$

$$M_{3,1} \cong \begin{bmatrix} \pi\varepsilon_1 & 1 & 1 \\ 1 & \overline{\pi\varepsilon_1} & 0 \\ 1 & 0 & \overline{\pi\varepsilon_1} \end{bmatrix}, \quad \overline{M}_{3,1} \cong \begin{bmatrix} \overline{\pi\varepsilon_1} & 1 & 1 \\ 1 & \pi\varepsilon_1 & 0 \\ 1 & 0 & \pi\varepsilon_1 \end{bmatrix}.$$

K_3 appears when $m = 13$. disc$(K_3) = \varepsilon_1$, disc$(M_{3,0}) = \overline{\pi}$, disc$(\overline{M}_{3,0}) = \pi$, disc$(M_{3,1}) = \overline{\pi\varepsilon_1}$, disc$(\overline{M}_{3,1}) = \pi\varepsilon_1$. *All isometric relations are:*

$$M_{3,0} \cong M_{3,1} \Longleftrightarrow \overline{M}_{3,0} \cong \overline{M}_{3,1} \Longleftrightarrow m \in M_3^2 \text{ with } \varepsilon_1 = \varepsilon_0^2,$$

$$M_{3,0} \cong \overline{M}_{3,1}, \Longleftrightarrow \overline{M}_{3,0} \cong M_{3,1} \Longleftrightarrow m \in M_3^1.$$

Let $n = 4$ and $\{x_1, x_2, x_3, x_4\}$ be a minimal N3-system for an indecomposable N3-lattice L. By Proposition 3.3 we may assume that L is isometric to

$$\begin{bmatrix} \alpha & 1 & 1 & 0 \\ 1 & \overline{\alpha} & 0 & 0 \\ 1 & 0 & \overline{\alpha} & 1 \\ 0 & 0 & 1 & \alpha \end{bmatrix}, \begin{bmatrix} \alpha & 1 & 1 & 1 \\ 1 & \overline{\alpha} & 0 & 0 \\ 1 & 0 & \overline{\alpha} & 0 \\ 1 & 0 & 0 & \overline{\alpha} \end{bmatrix}, \begin{bmatrix} \alpha & 1 & 1 & 0 \\ 1 & \overline{\alpha} & 0 & 1 \\ 1 & 0 & \overline{\alpha} & 1 \\ 0 & 1 & 1 & \alpha \end{bmatrix}, \begin{bmatrix} \alpha & 1 & 1 & 0 \\ 1 & \overline{\alpha} & 0 & -1 \\ 1 & 0 & \overline{\alpha} & 1 \\ 0 & -1 & 1 & \alpha \end{bmatrix},$$

with $\alpha \in \{\pi, \overline{\pi}, \pi\varepsilon_1, \overline{\pi\varepsilon_1}\}$. We delete the third lattice because of the discriminant $= -3$. We also delete the second lattice since Q $(\overline{\alpha}x_1 - x_2 - x_3 - x_4) = 0$ or $x_4 = \overline{\alpha}x_1 - x_2 - x_3$, which contradicts the minimality of the N3-system. We note that the first lattice $= \langle x_1, x_2, x_3, x_4 \rangle = \langle x_1, \overline{\alpha}x_1 - x_2 - x_3, x_3, x_4 \rangle \cong$ the fourth lattice.

For the first lattice we may take $\alpha = \pi$ or $\pi\varepsilon_1$. We write them as $M_{4,0}$ and $M_{4,1}$. We can see that $M_{4,0}$ and $M_{4,1}$ are isometric if and only if $\varepsilon_1 = \varepsilon_0^2$ by a simple computation.

Proposition 3. 4. *Suppose $n = 4$ and $m \in M_3$. Then L is isometric to one of the following:*

$$M_{4,0} \cong \begin{bmatrix} \pi & 1 & 0 & 0 \\ 1 & \overline{\pi} & 1 & 0 \\ 0 & 1 & \pi & 1 \\ 0 & 0 & 1 & \overline{\pi} \end{bmatrix}, \quad M_{4,1} \cong \begin{bmatrix} \overline{\pi\varepsilon_1} & 1 & 0 & 0 \\ 1 & \pi\varepsilon_1 & 1 & 0 \\ 0 & 1 & \overline{\pi\varepsilon_1} & 1 \\ 0 & 0 & 1 & \pi\varepsilon_1 \end{bmatrix},$$

where $\mathrm{disc}(M_{4,0}) = \mathrm{disc}(M_{4,1}) = 1$. $M_{4,0} \cong M_{4,1}$ *if and only if* $\varepsilon_1 = \varepsilon_0^2$.

Theorem 3. 1. *All indecomposable $N3$-lattices are given in Propositions3.1, 3.2, 3.3 and 3.4.*

Proof This is a consequence from Lemma 1. 2, Proposition 1. 2 and the fact that the lattices in Proposition 3. 4 are unimodular. In fact, for N3-lattice L with a minimal system $\{x_1, \ldots, x_n\}$, we can assume that the system satisfies the property in Proposition 1. 2. If $n > 4$, then we see that the lattice $L' = \langle x_1, \ldots, x_4 \rangle$ is unimodular by Proposition 3. 4. Hence we have $L = L' \perp L''$ for some lattice L'' by Lemma 1. 2, which contradicts the indecomposability of L. □

REFERENCES

1. M. Ozeki., *Note on the positive definite integral quadratic lattices,* Jour. Math. Soc. Japan, 28(1976), 421-446.
2. Y. Mimura., *On 2-lattices over real quadratic integers,* Mathematics Seminar Notes, 7 (1979), 327-342.
3. Y. Mimura., *On 2-lattices in an hermitian space,* Mathematica Japonica, 27(1981), 213-224.
4. O. T. O 'Meara., *Introduction to quadratic forms,* Springer-Verlag, 1962.